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STABILITY AND SIMULATION OF MEASLES TRANSMISSION MODEL WITH AND WITHOUT VACCINATION

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Abstract

To figure out the dynamics of the mathematical model of measles transmission with and without vaccination as a preventive measure is the purpose of this paper. The studies carried out include the stability analysis of the model, and using a computational program, the simulation is performed to synchronize the analytical results. The comparison between the stability of measles transmission model with vaccination and without vaccination is obtained in this paper. Two stationary points are obtained in each population. In terms of population with vaccination,

$$T_1 = (1, 0) \text{ and } T_2 = \left(\frac{\mu + \alpha}{\beta}, -\frac{\mu(-\beta + \mu + \alpha)}{\beta(\mu + \alpha)}\right)$$

are the stationary points. On the other hand, in population without vaccination,

$$T_3 = (1 - \varepsilon, 0) \text{ and } T_4 = \left(\frac{\mu + \alpha}{\beta}, -\frac{\mu(-(1 - \varepsilon)\beta + \mu + \alpha)}{\beta(\mu + \alpha)}\right)$$

are the stationary points. We found T_1 stable for $\beta < \mu + \alpha,$ and T_2

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stable for $\beta > \mu + \alpha$. In the second population, the model would be stable around T_3 and T_4 when $\beta < \frac{\mu + \alpha}{1 - \varepsilon}$ and $\beta > \frac{\mu + \alpha}{1 - \varepsilon}$, respectively.

Background

It is well known that mathematics and biology are well interrelated. The biological interpretation can be quite helpful in guessing identities or estimates and even in suggesting quick and elegant proofs (Diekmann et al. [7]). Mathematical model became a tool.

One of real life problems is disease transmission, which be death affected or not. Examples of non-fatal diseases are measles, influenza, and other such. Duncan et al. [3] conducted research about scarlet fever death in Liverpool and they found that the system was oscillatory with clear epidemics on a basic endemic level. They used SEIR model.

This paper discusses about the transmission of measles disease using SIR model. Two populations are considered: population with vaccination as a preventive measure and population without vaccination, with assumption that both of populations are constant and there is no migration from or into the population.

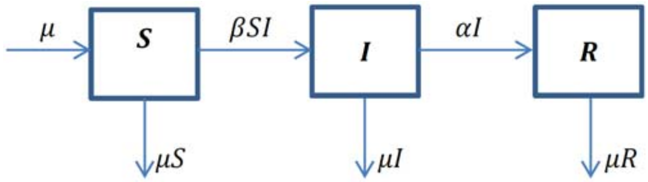
SIR Model

The transmission of measles disease in a population and in a period time can be modelled by the SIR model. Suppose in a certain time t, the population consists of:

- $S(t)$, *susceptible*: a subpopulation of those members who are susceptible to the disease,
- $I(t)$, *infective*: a subpopulation of those members who have contacted the disease,
- $R(t)$, *recovered*: people who have been cured of the disease,

with the proportion of $S + I + R = 1$.

In terms of measles transmission without vaccination, the compartment we put together is as follows:



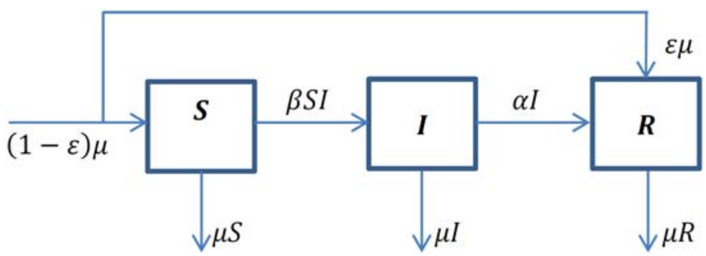
The mathematical model of the diagram is

$$\frac{dS}{dt} = \mu - \mu S - \beta SI,$$

$$\frac{dI}{dt} = \beta SI - \mu I - \alpha I,$$

$$\frac{dR}{dt} = \alpha I - \mu R. \tag{1}$$

The following diagram is the compartment of measles transmission in population with preventive vaccine:



The mathematical model of the diagram is

$$\frac{dS}{dt} = (1 - \varepsilon)\mu - \mu S - \beta SI,$$

$$\frac{dI}{dt} = \beta SI - \mu I - \alpha I,$$

$$\frac{dR}{dt} = \varepsilon\mu + \alpha I - \mu R, \tag{2}$$

(Yıldırım and Cherruault [11]), where

S is the number of vulnerable population of the measles,

I is the number of the infected population,

R is a population recovering from measles,

μ is natality (positive number),

β is the rate of disease transmission (positive number),

α is the rate of recovery (positive number),

ε is the rate of vaccination between S and R (positive number).

Equilibrium Points

To obtain the equilibrium points of a system, the right-hand side of the system must be equal to zero (Henner et al. [6]).

From system (1), we obtain two equilibria (S, I), which are $T_1 = (1, 0)$ and $T_2 = \left(\frac{\mu + \alpha}{\beta}, -\frac{\mu(-\beta + \mu + \alpha)}{\beta(\mu + \alpha)}\right)$.

From system (2), we also obtain two equilibria (S, I), which are $T_3 = (1 - \varepsilon, 0)$ and $T_4 = \left(\frac{\mu + \alpha}{\beta}, -\frac{\mu(-(1-\varepsilon)\beta + \mu + \alpha)}{\beta(\mu + \alpha)}\right)$.

Stability Analysis

To do the stability analysis of system (1), a linearization is needed. For instance,

$$f_1(S, I, R) = \mu - \mu S - \beta SI,$$

$$g_1(S, I, R) = \beta SI - \mu I - \alpha I.$$

Then, the Jacobian matrix is

$$J = \begin{pmatrix} -\mu - \beta I & -\beta S \\ \beta I & \beta S - \mu - \alpha \end{pmatrix}.$$

Theorem 1. *System* (1) *is stable at the point* $T_1 = (1, 0)$ *for* $\beta < \mu + \alpha$.

Proof. For the point $T_1 = (1, 0)$, the Jacobian matrix is given as follows:

$$J = \begin{pmatrix} -\mu & -\beta \\ 0 & \beta - \mu - \alpha \end{pmatrix}.$$

We obtain the eigenvalues $\lambda_1 = -\mu$ or $\lambda_2 = \beta - \mu - \alpha$.

To make the system stable, we should have $\beta < \mu + \alpha$.

Taking $\mu = 0.02$, $\alpha = 0.1$, and $\beta = 0.08$, we obtain the equilibrium point $T_1 = (1, 0)$. The behavior of system (1) is shown in Figures 1 and 3.

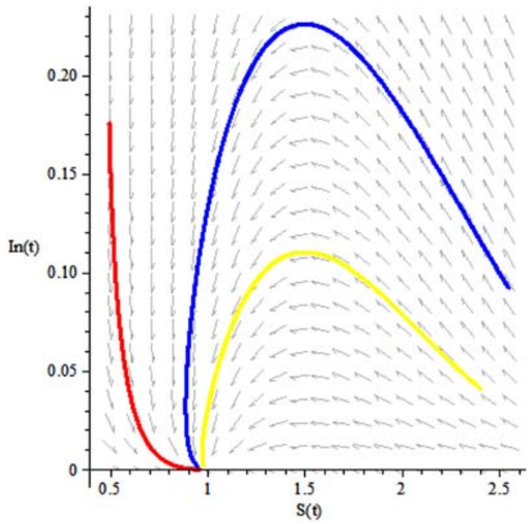


Figure 1

Theorem 2. *System* (1) *is stable at the point*

$$T_2 = \left(\frac{\mu + \alpha}{\beta}, -\frac{\mu(-\beta + \mu + \alpha)}{\beta(\mu + \alpha)}\right) \text{ for } \beta > \mu + \alpha.$$

Proof. For the point $T_2 = \left(\frac{\mu + \alpha}{\beta}, -\frac{\mu(-\beta + \mu + \alpha)}{\beta(\mu + \alpha)}\right)$, the Jacobian matrix is

$$J = \begin{pmatrix} -\mu + \dfrac{\mu(-\beta + \mu + \alpha)}{(\mu + \alpha)} & -(\mu + \alpha) \\ -\dfrac{\mu(-\beta + \mu + \alpha)}{(\mu + \alpha)} & 0 \end{pmatrix},$$

(Anton and Rorres [1]).

We obtain two eigenvalues,

$$\lambda_{3,4} = \frac{1}{2}\left(-\frac{\mu\beta}{\mu + \alpha} \pm \sqrt{\left(\frac{\mu\beta}{\mu + \alpha}\right)^2 - 4\mu(\beta - \alpha - \mu)}\right),$$

that is

(i) $\lambda_3 = \frac{1}{2}\left(-\frac{\mu\beta}{\mu + \alpha} + \sqrt{\left(\frac{\mu\beta}{\mu + \alpha}\right)^2 - 4\mu(\beta - \alpha - \mu)}\right)$ which is negative

or the real part is negative for $4\mu(\beta - \alpha - \mu) > 0 \leftrightarrow \beta > \mu + \alpha$.

(ii) $\lambda_4 = \frac{1}{2}\left(-\frac{\mu\beta}{\mu + \alpha} - \sqrt{\left(\frac{\mu\beta}{\mu + \alpha}\right)^2 - 4\mu(\beta - \alpha - \mu)}\right)$ which is negative

or the real part is negative for all positive parameters.

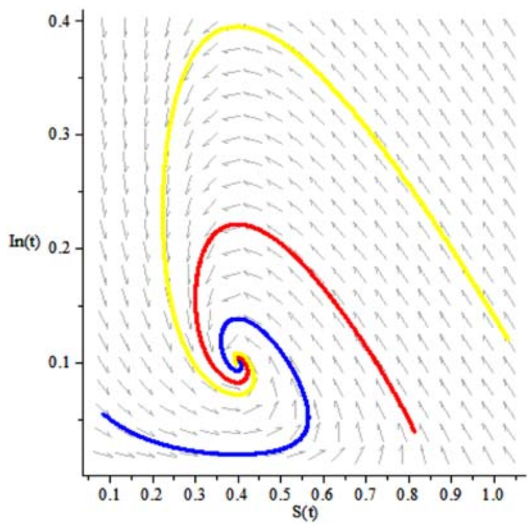


Figure 2

Taking $\mu = 0.02$, $\alpha = 0.1$, and $\beta = 0.3$, we obtain the equilibrium point $T_2 = (0.4, 0.1)$. The behavior of system (1) is shown in Figures 2 and 4.

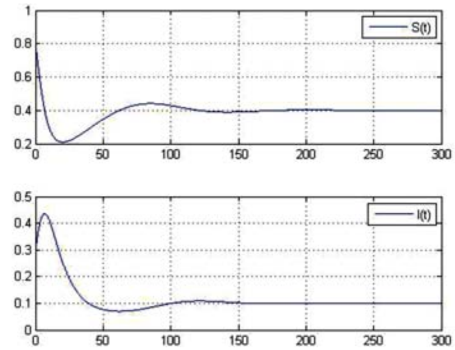


Figure 3

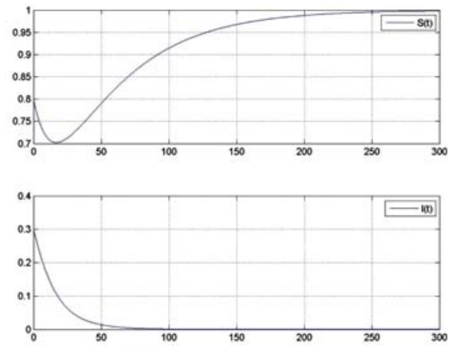


Figure 4

To do the stability analysis of system (2), a linearization is needed. For instance,

$$f_2(S, I, R) = (1 - \varepsilon)\mu - \mu S - \beta SI,$$

$$g_2(S, I, R) = \beta SI - \mu I - \alpha I.$$

Then, the Jacobian matrix is

$$J = \begin{pmatrix} -\mu - \beta I & -\beta S \\ \beta I & \beta S - \mu - \alpha \end{pmatrix}.$$

Theorem 3. *System* (2) *is stable at the point* $T_3 = (1 - \varepsilon, 0)$ *for* $\beta < \frac{\mu + \alpha}{1 - \varepsilon}$.

Proof. For the point $T_3 = (1 - \varepsilon, 0)$, the Jacobian matrix is given as follows:

$$J = \begin{pmatrix} -\mu & -\beta(1 - \varepsilon) \\ 0 & \beta(1 - \varepsilon) - \mu - \alpha \end{pmatrix}.$$

We obtain $\lambda_5 = -\mu$ or $\lambda_6 = \beta(1 - \varepsilon) - \mu - \alpha$.

To make the system stable, we should have $\beta < \frac{\mu + \alpha}{1 - \varepsilon}$.

Taking $\mu = 0.02$, $\alpha = 0.1$, $\varepsilon = 0.3$ and $\beta = 0.08$, we obtain the equilibrium point $T_3 = (0.7, 0)$. The behavior of system (2) is shown in Figures 5 and 7.

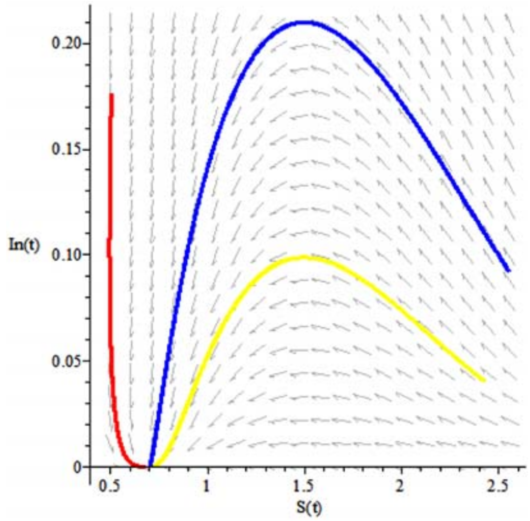


Figure 5

Theorem 4. *System* (2) *is stable at the point*

$$T_4 = \left(\frac{\mu+\alpha}{\beta}, -\frac{\mu(-(1-\varepsilon)\beta+\mu+\alpha)}{\beta(\mu+\alpha)}\right) \textit{ for } \beta > \frac{\mu+\alpha}{1-\varepsilon}.$$

Proof. For the point $T_4 = \left(\frac{\mu+\alpha}{\beta}, -\frac{\mu(-(1-\varepsilon)\beta+\mu+\alpha)}{\beta(\mu+\alpha)}\right)$, the Jacobian matrix is

$$J = \begin{pmatrix} -\mu + \dfrac{\mu(-(1-\varepsilon)\beta+\mu+\alpha)}{(\mu+\alpha)} & -(\mu+\alpha) \\ -\dfrac{\mu(-(1-\varepsilon)\beta+\mu+\alpha)}{(\mu+\alpha)} & 0 \end{pmatrix}.$$

We have

$$\lambda_{7,8} = \frac{1}{2}\left(-\frac{(1-\varepsilon)\mu\beta}{\mu+\alpha} \pm \sqrt{\left(\frac{(1-\varepsilon)\mu\beta}{\mu+\alpha}\right)^2 - 4\mu((1-\varepsilon)\beta-\alpha-\mu)}\right),$$

that is

(i) $\lambda_7 = \frac{1}{2}\left(-\frac{(1-\varepsilon)\mu\beta}{\mu+\alpha} + \sqrt{\left(\frac{(1-\varepsilon)\mu\beta}{\mu+\alpha}\right)^2 - 4\mu((1-\varepsilon)\beta-\alpha-\mu)}\right)$, which

is negative or the real part is negative for $4\mu((1-\varepsilon)\beta - \alpha - \mu) > 0 \leftrightarrow \beta > \frac{\mu+\alpha}{1-\varepsilon}$,

(ii) $\lambda_8 = \frac{1}{2}\left(-\frac{(1-\varepsilon)\mu\beta}{\mu+\alpha} - \sqrt{\left(\frac{(1-\varepsilon)\mu\beta}{\mu+\alpha}\right)^2 - 4\mu((1-\varepsilon)\beta-\alpha-\mu)}\right)$, which

is negative or the real part is negative for all positive parameters.

Taking $\mu = 0.02$, $\alpha = 0.1$, $\varepsilon = 0.3$, and $\beta = 0.3$, we obtain the equilibrium point $T_4 = (0.4, 0.05)$. The behavior of system (2) is shown in Figures 6 and 8.

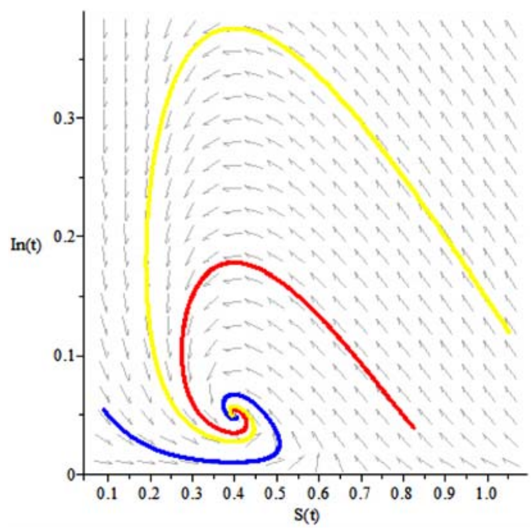


Figure 6

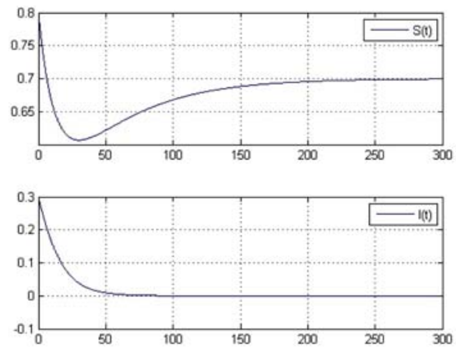


Figure 7

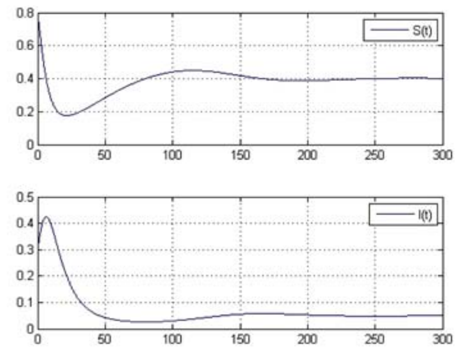


Figure 8

Conclusion

Two equilibria required in the population without vaccination are $T_1 = (1, 0)$ and $T_2 = \left(\frac{\mu + \alpha}{\beta}, -\frac{\mu(-\beta + \mu + \alpha)}{\beta(\mu + \alpha)}\right)$. In terms of population with preventive vaccine, two equilibria,

$$T_3 = (1 - \varepsilon, 0) \text{ and } T_4 = \left(\frac{\mu + \alpha}{\beta}, -\frac{\mu(-(1-\varepsilon)\beta + \mu + \alpha)}{\beta(\mu + \alpha)}\right)$$

are obtained. Points T_1 and T_3 are called *disease-free equilibria* for each model. It is because the infected population I is 0. Points T_3 and T_4 are called *endemic equilibria* for each model because the infected population exists.

The stability of the disease-free equilibria T_1 and T_3 occurs if $\beta < \mu + \alpha$ and $\beta > \mu + \alpha$, respectively. The stability of T_3 and T_4 occurs when $\beta < \frac{\mu + \alpha}{1 - \varepsilon}$ and $\beta > \frac{\mu + \alpha}{1 - \varepsilon}$, respectively. The stability of the endemic equilibrium holds for all the positive parameter values.

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